TITLE:

Boundary value problem for Hyperfunction solutions to Fuchsian systems (Algebraic analytic methods in complex partial differential equations)

AUTHOR(S):

Yamazaki, Susumu

CITATION:

Yamazaki, Susumu. Boundary value problem for Hyperfunction solutions to Fuchsian systems (Algebraic analytic methods in complex partial differential equations). 数理解析研究所講究録 2017, 2020: 96-113

ISSUE DATE:

2017-04

URL:

http://hdl.handle.net/2433/231743

RIGHT:

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数理解析研究所講究録
第2020巻 2017年 96-113

Boundary value problem for Hyperfunction solutions to Fuchsian systems

By

Susumu YAMAZAKI*

Abstract

In the framework of algebraic analysis, a general boundary value morphism is defined for any hyperfunction solutions to the Fuchsian system of analytic linear partial differential equations in derived category, and the injectivity of this morphism in zero-th cohomologies (that is, the Holmgren type theorem) is proved. Moreover, under a kind of hyperbolicity condition, it is proved that this morphism is surjective (that is, the solvability). These results extend that of H. Tahara and Laurent-Monteiro Fernandes to general Fuchsian systems.

Introduction

In this article, we announce of results about boundary value problems for hyperfunction solutions along an initial boundary to the *Fuchsian system* of analytic linear differential equations in the framework of *Algebraic Analysis.*

Fuchsian partial differential operator was first defined by Baouendi-Goulaouic [1], and Tahara [22] defined a *Fuchsian Volevič system* as a generalization of Fuchsian partial differential operator. Moreover Laurent-Monteiro Fernandes [10] defined a Fuchsian $\mathscr{D}_X$-Module. Here and in what follows, we shall write a *Ring* or a *Module* etc. with capital letters, instead of a *sheaf of rings* or a *sheaf of left modules* etc. We remark that the notion of Fuchsian $\mathscr{D}_X$-Modules includes Fuchsian Volevič systems.

For Cauchy problem in the framework of hyperfunctions on the real domain, we refer to Tahara [22], Oaku [16] and Oaku-Yamazaki [19] and Yamazaki [24]. For a boundary value problems for hyperfunction solutions, Laurent-Monteiro Fernandes [11]

2010 Mathematics Subject Classification(s): Primary 35A27; Secondary 32C38, 35L57, 58J15.
Key Words: Boundary value problem, Fuchsian system, hyperfunctions, Algebraic analysis.
This research was partially supported by the Grant-in-Aid for Scientific Research (C) 15K04908, Japan Society for the Promotion of Science.
*Department of General Education, College of Science and Technology, Nihon University, 24–1 Narashinodai 7-chome, Funabashi-shi, Chiba 274–8501, Japan.

give a general framework, and using results of [9], for any regular-specializable system (i.e. Fuchsian with constant characteristic exponents case), they defined an injective boundary value morphism (see also [14], [15]), and discussed solvability. For a microlocal counterpart, see Yamazaki [23].

In this paper, along the line of [11] and [23], we shall define an injective boundary value morphism for hyperfunction solutions to general Fuchsian system and state the unique solvability theorem for the boundary value problem in the category of hyperfunctions. For this purpose, by using precise analysis due to Tahara [22] and an idea of Oaku [18], we shall define a sort of nearby cycles for general Fuchsian Modules.

The contents of this article are appeared in RIMS Kôkyûroku Bessatsu **B57**, and details will be appeared in a forthcoming paper [25].

§1. Preliminaries

In this section, we shall fix the notation and recall known results used in later sections. Our main reference is Kashiwara-Schapira [7].

We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of all the integers, real numbers and complex numbers respectively. Moreover we set $\mathbb{N} := \{n \in \mathbb{Z}; n \geqslant 1\} \subset \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}^+ := \{r \in \mathbb{R}; r > 0\}$ and $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$.

In this paper, all the manifolds are assumed to be *paracompact*. Let Z be a manifold. For a subset $A \subset Z$, we denote by $\operatorname{Int} A$ and $\operatorname{Cl} A$ the interior and the closure of A respectively. Let $\mathcal{A}$ be a Ring on Z. We denote by $\mathcal{A}^{\mathrm{op}}$ the opposed Ring, and we regard right $\mathcal{A}$-Modules as (left) $\mathcal{A}^{\mathrm{op}}$-Modules. We denote by $\mathfrak{Mod}(\mathcal{A})$ the category of $\mathcal{A}$-Modules, and by $\mathfrak{Coh}(\mathcal{A})$ the full subcategory of $\mathfrak{Mod}(\mathcal{A})$ consisting of coherent $\mathcal{A}$-Modules. Further we denote by $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ the bounded derived category of complexes of $\mathcal{A}$-Modules, and by $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A})$ the full subcategory of $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ consisting of objects with coherent cohomologies. We set $\mathbf{D}^{\mathrm{b}}(Z) := \mathbf{D}^{\mathrm{b}}(\mathbb{C}_Z)$ etc. for short. Set $* \otimes * := * \underset{\mathbb{C}_Z}{\otimes} *$ etc. We denote by $\mathscr{or}_Z$ the orientation sheaf. Let $f\colon W \to Z$ be a continuous mapping between manifolds. Then the relative orientation sheaf is defined by $\mathscr{or}_{W/Z} := \mathscr{or}_W \otimes f^{-1}\mathscr{or}_Z$. Further $\omega_{W/Z} = \mathscr{or}_{W/Z}[\dim W - \dim Z]$ denotes the dualizing complex, and $\omega^{\otimes -1}_{W/Z} = \mathscr{or}_{W/Z}[\dim Z - \dim W]$ its dual. If $\tau\colon E \to Z$ is a vector bundle over a manifold Z, we set $\dot{E} := E \setminus Z$ and $\dot{\tau}$ the restriction of τ to $\dot{E}$. Let $\pi\colon E^* \to Z$ the dual bundle.

Let $\mathcal{F}$ be an object of $\mathbf{D}^{\mathrm{b}}(Z)$, and $T^*Z \to Z$ the cotangent bundle of Z. We denote by $\mathrm{SS}(\mathcal{F})$ the *microsupport* of $\mathcal{F}$ due to Kashiwara-Schapira (see [7]). $\mathrm{SS}(\mathcal{F})$ is a closed conic involutive subset of T^*Z and described as follows: Let $\mathring{p}$ be a point of T^*Z. Then $\mathring{p} \notin \mathrm{SS}(\mathcal{F})$ if the following condition holds: there exists a neighborhood U of $\mathring{p}$ in T^*Z such that for any $\mathring{z} \in Z$ and any real valued real analytic function ψ defined on a sufficiently small neighborhood of $\mathring{z}$ satisfying $(\mathring{z}; d\psi(\mathring{z})) \in U$, it follows that

$$\boldsymbol{R}\Gamma_{\{z;\, \psi(z) \geqslant \psi(\mathring{z})\}}(\mathcal{F})_{\mathring{z}} = 0.$$

Note that $\mathrm{SS}(\mathcal{F}) \cap T^*_Z Z = \operatorname{supp} \mathcal{F}$.

Next, let Z be a complex manifold with a local coordinate system $z = x + \sqrt{-1}\, y$, we use the following identifications as in [20, Chapter I]:

$$\begin{aligned} TZ \ni (z; \langle v, \partial_z \rangle) &\leftrightarrow (x, y; \langle \operatorname{Re} v, \partial_x \rangle + \langle \operatorname{Im} v, \partial_y \rangle) \in TZ^{\mathbb{R}}, \\ T^*Z \ni (z; \langle \zeta, dz \rangle) &\leftrightarrow (x, y; \langle \operatorname{Re} \zeta, dx \rangle - \langle \operatorname{Im} \zeta, dy \rangle) \in T^*Z^{\mathbb{R}}, \end{aligned}$$

where $Z^{\mathbb{R}}$ denotes the underlying real manifold of Z. Thus, for the complex dual inner product $\langle *, * \rangle \colon TZ \underset{Z}{\times} T^*Z \to \mathbb{C}$, the corresponding real dual inner product is $\operatorname{Re}\langle *, * \rangle \colon TZ \underset{Z}{\times} T^*Z \to \mathbb{R}$.

Let M be an $(n+1)$-dimensional real analytic manifold and N a one-codimensional closed real analytic submanifold of M. Let X and Y be complexifications of M and N respectively such that Y is a closed submanifold of X and that $Y \cap M = N$. Let $\tilde{z} = \tilde{x} + \sqrt{-1}\, \tilde{y}$ be a local coordinate system of X such that $\tilde{x}$ is a local coordinate system of M. We assume that there exists a $(2n+1)$-dimensional real analytic submanifold L of X containing both M and Y such that the triplet (N, M, L) is locally isomorphic to the triplet $(\{(x, 0) \in \mathbb{R}^n \times \{0\}\}, \{(x, t) \in \mathbb{R}^{n+1}\}, \{(z, t) \in \mathbb{C}^n \times \mathbb{R}\})$ by a local coordinate system $\tilde{z} = (z, \tau)$ with $\tilde{x} = (x_1, \ldots, x_n, t) = (x, t)$, $z = x + \sqrt{-1}\, y$ and $\tau = t + \sqrt{-1}\, s$ around each point of N (i.e. L is a partial complexification). We say such a local coordinate system *admissible*, and under this local coordinate system, we have:

$$(1.1)\qquad \begin{tikzcd} N = \mathbb{R}^n_x \times \{0\} \arrow[r, hook, "f_N"] \arrow[d, "\iota_N"] & M = \mathbb{R}^n_x \times \mathbb{R}_t \arrow[d, "\iota_M"] \arrow[dr, hook, "\iota"] & \\ Y = \mathbb{C}^n_z \times \{0\} \arrow[r, hook, "g"] \arrow[rr, bend right, "f"'] & L = \mathbb{C}^n_z \times \mathbb{R}_t \arrow[r, hook, "h"] & X = \mathbb{C}^n_z \times \mathbb{C}_\tau . \end{tikzcd}$$

Then we identify $\mathit{or}_{N/Y}$ with $f_N^{-1} \mathit{or}_{M/L}$. Let $\tau_N \colon T_N M \to N$ and $\pi_N \colon T^*_N M \to N$ be the *normal* and the *conormal bundles* to N in M respectively. By an admissible local coordinate system, we often identify normal bundles with base spaces; for example, $T_Y X = X$, $T_M X = X$, $T_N M = M$ etc. (i.e. we identify $(x; t) \in T_N M$ with $(x, t) \in M$). We denote by

$$(1.2)\qquad \begin{aligned} (\tilde{z}; \tilde{z}^*) = (z, \tau; z^*, \tau^*) &= (\tilde{x} + \sqrt{-1}\, \tilde{y}; \tilde{x}^* + \sqrt{-1}\, \tilde{y}^*) \\ &= (x + \sqrt{-1}\, y, t + \sqrt{-1}\, s; x^* + \sqrt{-1}\, y^*, t^* + \sqrt{-1}\, s^*) \end{aligned}$$

the associated local coordinate system of T^*X with the local coordinate system in (1.1).

The mapping f induces mappings:

$$\begin{tikzcd}
 & & N \arrow[r, hook, "f_N"] \arrow[dd, hook] \arrow[dl, hook'] \arrow[dr, phantom, "\square"] & M \arrow[dd, hook, "i_M"] \\
 & \sqrt{-1}\,T^*_NM \arrow[dl, two heads, "\pi"'] \arrow[dr, hook, "i"] & & \\
N \arrow[dr, "i_N"'] \arrow[rr, phantom, "\square"] & & N \underset{M}{\times} T^*_MX \arrow[r, hook, "f_{N\pi}"] \arrow[dl, two heads, "f_{Nd}"'] \arrow[dd, two heads, "\pi"] \arrow[dr, phantom, "\square"] & T^*_MX \arrow[dd, two heads, "\pi_M"] \\
 & T^*_NY \arrow[dr, two heads, "\pi_N"'] & & \\
 & & N \arrow[r, hook, "f_N"] & M
\end{tikzcd}$$

where π_N, π_M and π are canonical projections, i_N, i_M and i are zero-section embeddings, and $\square$ means that the square is *Cartesian.* Assume that $N = \varphi^{-1}(0)$ for an analytic function φ such that we may choose that $\varphi(\tilde{x}) = t$. We use the same symbol $\varphi\colon X \to \mathbb{C}$ to stand for the complexification, and we may assume that $\varphi(\tilde{z}) = \tau$. Then $d\varphi$ induces $\tilde{\varphi}\colon T_YX \to \mathbb{C}$, and we denote by $\hat{\sigma}\colon Y \to \dot{T}_YX$ the section of $T_YX \to \mathbb{C}$ given by $\tilde{\varphi}^{-1}(1)$, and by ${}^*\hat{\sigma}\colon Y \to \dot{T}^*_YX$ the section of $T^*_YX \to \mathbb{C}$ given by $d\varphi$. In the same way, $d\varphi$ induces $\tilde{\varphi}\colon T_NM \to \mathbb{R}$, and we can define mappings $\hat{s}\colon N \to \dot{T}_NM$ and ${}^*\hat{s}\colon N \to \sqrt{-1}\,\dot{T}^*_NM = \dot{T}^*_MX \cap \dot{T}^*_YX$. Under the local coordinate system in (1.1), we have

$$\hat{\sigma}(z) = (z, 1), \quad {}^*\hat{\sigma}(z) = (z; 1 \cdot d\tau), \quad \hat{s}(x) = (x, 1), \quad {}^*\hat{s}(x) = (x; \sqrt{-1}\,dt).$$

We set

$$\begin{aligned}
&\dot{T}_NM^+ := \mathbb{R}^+\hat{s}(N) \simeq \{(x,t);\, t > 0\} \subset T_NM^+ := \dot{T}_NM^+ \cup T_NN \simeq \{(x,t);\, t \geqslant 0\},\\
&\dot{T}^*_NM^+ := \frac{1}{\sqrt{-1}}\,\mathbb{R}^+\,{}^*\hat{s}(N) \simeq \{(x;t^*);\, t^* > 0\}.
\end{aligned}$$

As usual, let ν_* and μ_* be *specialization* and *microlocalization functors* respectively. We write $M \setminus N = \Omega_+ \sqcup \Omega_-$, where each $\Omega_\pm$ is an open subset and $\partial\Omega_\pm = N$. We set $M_+ := \Omega_+ \sqcup N$. By an admissible local coordinate system, we can write

$$\Omega_+ = \{(x,t) \in M;\, t > 0\} \subset M_+ = \{(x,t) \in M;\, t \geqslant 0\}.$$

Next, we denote by $\widetilde{M}_N$ and $\widetilde{L}_Y$ the normal deformations of N and Y in M and L respectively and regard $\widetilde{M}_N$ as a closed submanifold of $\widetilde{L}_Y$. We have the following commutative diagram:

$$\begin{tikzcd}
T_NM \arrow[rr, hook, "s_M"] \arrow[dr, two heads, "\tau_N"] \arrow[dd, hook] & & \widetilde{M}_N \arrow[dr, two heads, "p_M"] \arrow[dd, hook, "\tilde{\iota}'_M"'] & & \Omega_M \arrow[ll, hook', "j_M"'] \arrow[dl, two heads, "\tilde{p}_M"'] \arrow[dd, hook, "\tilde{\iota}_M"'] & \\
 & N \arrow[rr, hook] \arrow[dd, hook, "\iota_N"'] & & M \arrow[rr, hook, "\iota"] \arrow[dd, hook, "\iota_M"'] & & X \arrow[dd, equal] \\
T_YL \arrow[rr, hook, "s_L"] \arrow[dr, two heads, "\tau_Y"'] & & \widetilde{L}_Y \arrow[dr, two heads, "p_L"] & & \Omega_L \arrow[ll, hook', "j_L"'] \arrow[dl, two heads, "\tilde{p}_L"'] & \\
 & Y \arrow[rr, hook, "g"] & & L \arrow[rr, hook, "h"] & & X.
\end{tikzcd}$$

Using an admissible local coordinate system, we can write:

$$p_L\colon \widetilde{L}_Y = \{(z,t;r);\ r\in\mathbb{R},\ (z,rt)\in L\} \ni (z,t;r)\mapsto (z,rt)\in L,$$
$$p_M\colon \widetilde{M}_N = \{(x,t;r);\ r\in\mathbb{R},\ (x,rt)\in M\} \ni (x,t;r)\mapsto (x,rt)\in M,$$

$$T_Y L = \widetilde{L}_Y\cap\{(z,t;r);\ r=0\},\qquad \Omega_L = \widetilde{L}_Y\cap\{(z,t;r);\ r>0\},$$
$$T_N M = \widetilde{M}_N\cap\{(x,t;r);\ r=0\},\quad \Omega_M = \widetilde{M}_N\cap\{(x,t;r);\ r>0\}.$$

The mappings $\tilde{\tau}\colon T_Y L\to Y$, $p_L\colon \widetilde{L}_Y\to L$, $s_L\colon T_Y L\to \widetilde{L}_Y$ and $g\colon Y\to L$ induce natural mappings:

$$\begin{tikzcd}
N\underset{M}{\times} T^*_M L \arrow[r, "g_{Nd}"'] \arrow[d, "g_{N\pi}"] & T^*_N Y & T_N M\underset{N}{\times} T^*_N Y \arrow[l, "\tilde{\tau}_\pi"'] \arrow[r, "\sim", "\tilde{\tau}_d"'] & T^*_{T_N M} T_Y L \\
T^*_M L & \widetilde{M}_N\underset{M}{\times} T^*_M L \arrow[l, "p_{L\pi}"'] \arrow[r, "\sim", "p_{Ld}"'] & T^*_{\widetilde{M}_N}\widetilde{L}_Y & T_N M\underset{\widetilde{M}_N}{\times} T^*_{\widetilde{M}_N}\widetilde{L}_Y\,, \arrow[l, "s_{L\pi}"'] \arrow[u, "s_{Ld}", "\wr"']
\end{tikzcd}$$

and by these mappings we use the following identifications:

$$T_N M\underset{N}{\times} T^*_N Y = T^*_{T_N M} T_Y L = T_N M\underset{\widetilde{M}_N}{\times} T^*_{\widetilde{M}_N}\widetilde{L}_Y,\quad \widetilde{M}_N\underset{M}{\times} T^*_M L = T^*_{\widetilde{M}_N}\widetilde{L}_Y,$$

and we denote by

$$\pi_{N|M}\colon T^*_{T_N M} T_Y L = T_N M\underset{N}{\times} T^*_N Y = T^*_{T_N M} T_Y L\to T_N M,$$
$$\pi_{N,M}\colon T^*_{\widetilde{M}_N}\widetilde{L}_Y = \widetilde{M}_N\underset{M}{\times} T^*_M L\to \widetilde{M}_N,$$

the natural projections. $T_Y L\setminus T_Y Y$ has two components with respect to its fiber. We denote by $\dot{T}_Y L^+$ one of them as $\dot{T}_N M^+ = \dot{T}_Y L^+\cap T_N M$ and represent by fixing a local coordinate system

$$\dot{T}_Y L^+ = \{(z,t)\in T_Y L;\ t>0\}$$

(in this case we choose $\varphi(\tilde{z})=\tau$). Define open embeddings i_+ and i_{N+} by:

$$\begin{tikzcd}
\dot{T}_Y L^+ \arrow[r, hook, "i_+"] & T_Y L \\
\dot{T}_N M^+ \arrow[r, hook, "i_{N+}"] \arrow[u, hook] & T_N M. \arrow[u, hook]
\end{tikzcd}$$

We regard $\dot{T}_N M^+\underset{N}{\times} T^*_N Y$ as an open set of $T^*_{T_N M} T_Y L$. Moreover i_+ induces mappings:

$$\begin{tikzcd}
T^*_{\dot{T}_N M^+}\dot{T}_Y L^+ & \dot{T}_N M^+\underset{T_N M}{\times} T^*_{T_N M} T_Y L \arrow[l, "\simeq"'] \arrow[r, hook, "i_{\pi+}"] \arrow[d, "\wr"] & T^*_{T_N M} T_Y L \arrow[d, "\wr"] \\
& \dot{T}_N M^+\underset{N}{\times} T^*_N Y \arrow[r, hook, "i_{N+}\times\mathbb{1}"] & T_N M\underset{N}{\times} T^*_N Y.
\end{tikzcd}$$

Hence we identify $T^*_{\dot{T}_N M^+}\dot{T}_Y L^+$ with $\dot{T}_N M^+ \underset{N}{\times} T^*_N Y$, and $i_{\pi+}$ with $i_{N+} \times \mathbb{1}$. We set

$$\tilde{\tau}_{\pi+} := \tilde{\tau}_\pi \circ i_{\pi+} \colon \dot{T}_N M^+ \underset{N}{\times} T^*_N Y \twoheadrightarrow T^*_N Y.$$

Next, we recall the definition of the near-hyperbolicity condition:

1.1. Definition ([11, Definition 1.3.1]). Let $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(X)$. Then we say that $\mathcal{F}$ is *near-hyperbolic* at $\mathring{x} \in N$ in the ϵdt-codirection ($\epsilon = \pm$) if there exist positive constants C and ε_1 such that

$$\begin{aligned}\mathrm{SS}(\mathcal{F}) \cap \{(z,\tau;z^*,\tau^*) \in T^*X;\ |z-\mathring{x}| < \varepsilon_1,\ |\tau| < \varepsilon_1, \epsilon t > 0\}\\ \subset \{(z,\tau;z^*,\tau^*) \in T^*X;\ |t^*| \leqslant C((|y|+|s|)|y^*|+|x^*|)\}\end{aligned}$$

holds by the local coordinate system $(z,\tau;z^*,\tau^*)$ of T^*X in (1.2).

§ 2. Operators of Infinite Order

We inherit the notation from the preceding section. For a set (or a sheaf) S with a suitable algebraic structure, we denote by $\mathrm{Mat}_{m,n}(S)$ the set of matrices of size $m \times n$ whose components belong to S. We set $\mathrm{Mat}_m(S) := \mathrm{Mat}_{m,m}(S)$, and denote by $\mathbb{1}_m$ the identity matrix of size m. For the theory of $\mathscr{D}$-Modules, we refer to Björk [2], Kashiwara [3]. We denote by $\mathscr{O}_X$ and $\mathscr{D}_X$ the Rings of *holomorphic functions* and *holomorphic partial differential operators* on X. Let Ω_X be the sheaf of the *holomorphic forms with maximal degree* on X, and $\Omega_X^{\otimes -1} := \mathscr{H}om_{\mathscr{O}_X}(\Omega_X, \mathscr{O}_X)$. Let $\mathscr{D}_{Y\to X} := \mathscr{O}_Y \underset{f^{-1}\mathscr{O}_X}{\otimes} f^{-1}\mathscr{D}_X$ and $\mathscr{D}_{X\leftarrow Y} := \Omega_Y \underset{\mathscr{O}_Y}{\otimes} \mathscr{D}_{Y\to X} \underset{f^{-1}\mathscr{O}_X}{\otimes} f^{-1}\Omega_X^{\otimes -1}$ be the *transfer* $(\mathscr{D}_Y \otimes f^{-1}\mathscr{D}_X^{\mathrm{op}})$-and $(f^{-1}\mathscr{D}_X^{\mathrm{op}} \otimes \mathscr{D}_Y)$*-Modules* associated with $f \colon Y \hookrightarrow X$ respectively. For any $\mathscr{N} \in \mathbf{D}^{\mathrm{b}}(\mathscr{D}_X)$, we denote by

$$\boldsymbol{D}f^*\mathscr{N} := \mathscr{D}_{Y\to X} \overset{\boldsymbol{L}}{\underset{f^{-1}\mathscr{D}_X}{\otimes}} f^{-1}\mathscr{N} = \mathscr{O}_Y \overset{\boldsymbol{L}}{\underset{f^{-1}\mathscr{O}_X}{\otimes}} f^{-1}\mathscr{N}, \quad \boldsymbol{D}f^!\mathscr{N} := \boldsymbol{D}_Y \boldsymbol{D}f^* \boldsymbol{D}_X \mathscr{N},$$

the *inverse image* and the *extraordinary inverse image* respectively in $\mathscr{D}$-Module theory. Here for a complex manifold Z and $\mathscr{L} \in \mathbf{D}^{\mathrm{b}}(\mathscr{D}_Z)$, we set

$$\boldsymbol{D}_Z\mathscr{L} := \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Z}(\mathscr{L}, \mathscr{D}_Z) \overset{\boldsymbol{L}}{\underset{\mathscr{O}_Z}{\otimes}} \Omega_Z^{\otimes -1}[\dim Z] \quad (\dim Z \text{ is the complex dimension of } Z).$$

Under the local coordinate system in (1.1), we set $\vartheta := \tau\partial_\tau$ (or $t\partial_t$ in real case).

2.1. Definition. Let $\mathscr{M} \in \mathfrak{Coh}(\mathscr{D}_X)$. We say that $\mathscr{M}$ is *near-hyperbolic* at $\mathring{x} \in N$ in the ϵdt-codirection if so is $\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ in the sense of Definition 1.1.

2.2. Definition. Let $m \in \mathbb{N}$ and $w \in \mathbb{N}_0$ with $w \leqslant m$. Then we say that P is a *Fuchsian partial differential operator* of weight (m, w) in the sense of Baouendi-Goulaouic [1] if P can be written in the following form:

$$P(z,\tau,\partial_z,\partial_\tau) = \tau^{m-w}\partial_\tau^m + \sum_{i=w}^{m-1} P_i(z,\tau,\partial_z)\,\tau^{i-w}\partial_\tau^i + \sum_{i=0}^{w-1} P_i(z,\tau,\partial_z)\,\partial_\tau^i\,,$$

where $P_i \in \mathscr{D}_{X(m-i)}$ with $[P_i, \tau] = 0$ $(0 \leqslant i \leqslant m)$, and $P_i(z, 0, \partial_z) \in \mathscr{O}_Y$ $(w \leqslant i \leqslant m)$.

We say that P is Fuchsian hyperbolic in the sense of Tahara [22] if the principal symbol is written as $\sigma_m(P)(z,\tau,z^*,\tau^*) = \tau^{m-w}p(z,\tau,z^*,\tau^*)$, and $p(z,\tau,z^*,\tau^*)$ satisfies the following:

$$\text{(2.1)}\quad \begin{cases} \text{If } (x,t;x^*) \text{ are real, all the roots to the equation } p(x,t,x^*,\tau^*) = 0 \text{ with respect} \\ \text{to } \tau^* \text{ are real.} \end{cases}$$

Then $\mathscr{D}_X/\mathscr{D}_X P$ is near-hyperbolic in the $\pm dt$-codirections (see [11, Lemma 1.3.2]).

Note that a Fuchsian partial differential operator of weight $(m, 0)$ is called *an operator with regular singularity along Y in a weak sense* in Kashiwara-Oshima [6], and if the weight of P is (m, m), then Y is non-characteristic for $\mathscr{D}_X/\mathscr{D}_X P$.

2.3. Definition. We call a matrix $P = \vartheta - A(z,\tau,\partial_z) \in \mathrm{Mat}_m(\mathscr{D}_X)$ is a *Fuchsian Volevič system* of size m due to Tahara [22] if the following hold: Let $A_{ij}(z,\tau,\partial_z)$ be the (i,j)-component of $A(z,\tau,\partial_z)$.

(1) There exists $\{n_i\}_{i=1}^m \subset \mathbb{Z}$ such that $A_{ij}(z,\tau,\partial_z) \leqslant \mathscr{D}_{X(n_i-n_j+1)}$ for any $1 \leqslant i,j \leqslant m$.
(2) $[A_{ij}, \tau] = 0$ and $A_{ij}(z, 0, \partial_z) \in \mathscr{O}_Y$ for any $1 \leqslant i, j \leqslant m$.
Moreover we say that P is Fuchsian hyperbolic in the sense of Tahara [22] if

$$\det[\tau\tau^*\mathbb{1}_m - \sigma(A)(z,\tau,z^*)] = \tau^m p(z,\tau,z^*,\tau^*),$$

and $p(z,\tau,z^*,\tau^*)$ satisfies the condition (2.1). Then $\mathscr{D}_X^m/\mathscr{D}_X^m P$ satisfies the near-hyperbolicity condition. Here we set $\sigma(A)(z,\tau,z^*) := (\sigma_{n_i-n_j+1}(A_{ij})(z,\tau,z^*))_{i,j=1}^m$

Let $\mathcal{F}_Y(\mathscr{D}_X) \subset \mathfrak{Coh}(\mathscr{D}_X)$ denote the subcategory of Fuchsian $\mathscr{D}_X$-Modules along Y due to Laurent-Monteiro Fernandes [10].

2.4. Example. (1) If P is a Fuchsian partial differential operator, we can see that $\mathscr{D}_X/\mathscr{D}_X P \in \mathcal{F}_Y(\mathscr{D}_X)$.
(2) If P is a Fuchsian Volevič system of size m, then $\mathscr{D}_X^m/\mathscr{D}_X^m P \in \mathcal{F}_Y(\mathscr{D}_X)$.

2.5. Proposition. *Let $\mathscr{M} \in \mathfrak{Coh}(\mathscr{D}_X)$. Then $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$ if and only if locally there exists an epimorphism $\bigoplus_{i=1}^{I} \mathscr{D}_X/\mathscr{D}_X P_i \twoheadrightarrow \mathscr{M}$, where each P_i is a Fuchsian differential operator with weight $(m_i, 0)$.*

2.6. Proposition. *Let $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ be an exact sequence in $\mathfrak{Coh}(\mathscr{D}_X)$. Then $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$ if and only if $\mathscr{M}', \mathscr{M}'' \in \mathcal{F}_Y(\mathscr{D}_X)$.*

2.7. Definition. We take the admissible local coordinate system in (1.1), and write $X \times X = \{(z, \tau, w, \tau')\}$ on a neighborhood of $Y \times Y = \{(z, 0, w, 0) \in X \times X\}$. We set (see [18])

$$\Delta_{X/Y} := \{(z, \tau, w, \tau') \in X \times X;\ \tau = \tau'\} = \{(z, w, \tau)\}.$$

Then we regard $Y \times Y$ as a closed subset of $\Delta_{X/Y}$. Let $\Delta_Y \subset Y \times Y$ be the diagonal set. We have closed embeddings

$$\begin{tikzcd} Y \arrow[r, hook, "\delta_Y"] \arrow[d, hook, "f"'] & Y \times Y \arrow[d, hook, "\delta_f"'] \arrow[dr, "f \times f"] & \\ X \arrow[r, hook, "\delta"] & \Delta_{X/Y} \arrow[r, hook, "\delta_{X/Y}"] & X \times X \end{tikzcd}$$

where $\delta\colon X \ni (z, \tau) \mapsto (z, z, \tau) \in \Delta_{X/Y}$, $\delta_{X/Y}\colon \Delta_{X/Y} \ni (z, w, \tau) \mapsto (z, \tau, w, \tau) \in X \times X$ etc.

2.8. Remark. Under the assumption of the existence of a partial complexification L, we can show that $\Delta_{X/Y}$ (resp. $\Delta_{X/Y} \cap (M \times M)$) does not depend on the choice of admissible local coordinate systems on a neighborhood of $Y \times Y$ (resp. $N \times N$).

We set $\mathscr{O}_{X\times X}^{(0,n+1)} := \mathscr{O}_{X\times X} \underset{q_2^{-1}\mathscr{O}_X}{\otimes} q_2^{-1}\Omega_X = \Omega_{X\times X} \underset{q_1^{-1}\mathscr{O}_X}{\otimes} q_1^{-1}\Omega_X^{\otimes -1}$, where $q_i\colon X \times X \to X$ is the i-th projection, and set $\mathscr{O}_{Y\times Y}^{(0,n)}$ in the same way. Further we set

$$\mathscr{O}_{\Delta_{X/Y}}^{(0,n)} := \Omega_{\Delta_{X/Y}} \underset{p_1^{-1}\mathscr{O}_X}{\otimes} p_1^{-1}\Omega_X^{\otimes -1},$$

where $p_1 := q_1 \circ \delta_{X/Y}\colon \Delta_{X/Y} \to X$. Under the admissible local coordinate system, we see that $\mathscr{O}_{X\times X}^{(0,n+1)} = \mathscr{O}_{X\times X}\, dw\, d\tau'$, $\mathscr{O}_{Y\times Y}^{(0,n)} = \mathscr{O}_{Y\times Y}\, dw$ and $\mathscr{O}_{\Delta_{X/Y}}^{(0,n)} = \mathscr{O}_{\Delta_{X/Y}}\, dw$, where $dw := dw_1 \wedge \cdots \wedge dw_n$ etc. Let $\Delta_X \subset X \times X$ be the diagonal set. Then

$$\mathscr{D}_X^{\infty} = H_{\Delta_X}^{n+1}(\mathscr{O}_{X\times X}^{(0,n+1)}) \simeq \boldsymbol{R}\Gamma_{\Delta_X}(\mathscr{O}_{X\times X}^{(0,n+1)})[n+1]$$

is the Ring on X of *holomorphic partial differential operators of infinite order.* By the tangent mapping $\delta'\colon T_Y X \hookrightarrow T_{Y\times Y}\Delta_{X/Y}$ of $\delta\colon X \hookrightarrow \Delta_{X/Y}$, we regard $T_Y X$ as a closed subset of $T_{Y\times Y}\Delta_{X/Y}$.

2.9. Theorem. *The object $\boldsymbol{R}\Gamma_{T_Y X}(\nu_{Y\times Y}(\boldsymbol{R}\Gamma_{\Delta_{X/Y}}(\mathscr{O}_{X\times X})))$ is concentrated in degree $n+1$.*

For the proof, we use the abstract edge of the wedge theorem due to Kashiwara (see [5]).

2.10. Definition. We define

$$\begin{aligned}\widehat{\mathscr{D}}^{\nu}_{T_Y X} &:= \boldsymbol{R}\Gamma_{T_Y X}(\nu_{Y\times Y}(\boldsymbol{R}\Gamma_{\Delta_{X/Y}}(\mathscr{O}^{(0,n+1)}_{X\times X})))[n+1]\\ &= H^n_{T_Y X}(\nu_{Y\times Y}(H^1_{\Delta_{X/Y}}(\mathscr{O}^{(0,n+1)}_{X\times X}))).\end{aligned}$$

2.11. Remark. Let $\mathring{p} = (\mathring{z}, \mathring{\tau}) \in T_Y X \simeq \mathbb{C}^n \times \mathbb{C}$. For ρ, $\delta > 0$, we set

$$\mathbb{D}_\rho(\mathring{z}) := \bigcap_{i=1}^{n} \{z \in \mathbb{C}^n;\ |z_i - \mathring{z}_i| < \rho\}, \quad \mathbb{B}_\delta := \{\tau \in \mathbb{C};\ |\tau| < \delta\}.$$

Then $P = P(z, \tau, \partial_z, \partial_\tau) = \sum_{\alpha, i} a_{\alpha,i}(z,\tau)\partial_z^\alpha \partial_\tau^i \in \widehat{\mathscr{D}}^{\nu}_{T_Y X, \mathring{p}}$ is given as follows:

(a) Assume that $\mathring{\tau} = 0$. Then there exist an open neighborhood V of $\mathring{z}$ in Y and $\delta > 0$ such that $a_{\alpha,i}(z,\tau) \in \Gamma(V \times \mathbb{D}_\delta; \mathscr{O}_X)$, and there exists a function $\mathbb{R}^+ \ni \varepsilon \mapsto \delta(\varepsilon) \in]\,0, \delta\,[$ satisfying the following: for any $Z \Subset V$ and ε, $\varepsilon_0 > 0$, there exists $C_{Z,\varepsilon,\varepsilon_0} > 0$ such that

$$\sup\{|a_{\alpha,i}(z,\tau)|;\ (z,\tau) \in Z \times \mathbb{D}_{\delta(\varepsilon)}\} \leqslant \frac{C_{Z,\varepsilon,\varepsilon_0}\varepsilon^{|\alpha|}\varepsilon_0^{\,i}}{\alpha!\, i!}.$$

(b) Assume that $\mathring{\tau} \neq 0$. Then there exist an open a neighborhood V of $\mathring{z}$ in Y and δ, $\rho > 0$ such that $a_{\alpha,i}(z,\tau) \in \Gamma(V \times S_{\delta,\rho}(\mathring{\tau}); \mathscr{O}_X)$, and there exists a function $\mathbb{R}^+ \ni \varepsilon \mapsto \delta(\varepsilon) \in]\,0, \delta\,[$ satisfying the following: for any $Z \Subset V$ and ε, $\varepsilon_0 > 0$ and $S' \Subset S_{\delta(\varepsilon),\rho}(\mathring{\tau})$, there exists $C_{Z,S',\varepsilon,\varepsilon_0} > 0$ such that

$$\sup\{|a_{\alpha,i}(z,\tau)|;\ (z,\tau) \in Z \times S'\} \leqslant \frac{C_{Z,S',\varepsilon,\varepsilon_0}\varepsilon^{|\alpha|}\varepsilon_0^{\,i}}{\alpha!\, i!}.$$

Set $\tau_{X,Y} := f \circ \tau_Y : T_Y X \to X$.

2.12. Remark. (1) $\widehat{\mathscr{D}}^{\nu}_{T_Y X}$ is a Ring with formal adjoints, and $\tau_{X,Y}^{-1}\mathscr{D}_X^\infty$ is a Subring of $\widehat{\mathscr{D}}^{\nu}_{T_Y X}$, compatible with formal adjoints.

(2) $\nu_Y(\mathscr{O}_X)$ is a $\widehat{\mathscr{D}}^{\nu}_{T_Y X}$-Module.

2.13. Definition (Tahara [22]). We take the admissible local coordinate system in (1.1). Let $\mathring{z} \in Y$. For $m \in \mathbb{N}$, we define $P(z,\tau,\partial_z) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha(z,\tau)\partial_z^\alpha \in \widehat{\mathscr{D}}_{X|Y,\mathring{z}}$ as follows:

(a) There exist ρ, $\delta_0 > 0$ such that $a_\alpha(z,\tau) \in \Gamma(\mathrm{Cl}[\mathbb{D}_\rho(\mathring{z}) \times \mathbb{B}_{\delta_0}]; \mathscr{O}_X)$,

(b) there exist A, $m > 0$ satisfying the following: for any $0 < \delta \leqslant \delta_0$, there exists $C_\delta > 0$ such that

$$\max\{|a_\alpha(z,\tau)|;\ (z,\tau) \in \mathrm{Cl}[\mathbb{D}_\rho(\mathring{z}) \times \mathbb{B}_\delta]\} \leqslant \frac{C_\delta(A\delta^{1/m})^{|\alpha|}}{\alpha!}.$$

We can see that $\widehat{\mathscr{D}}_{X|Y,\tau_Y(\mathring{p})} \subset \widehat{\mathscr{D}}^{\nu}_{T_Y X, \tau_Y(\mathring{p})} \subset \widehat{\mathscr{D}}^{\nu}_{T_Y X, \mathring{p}}$ for any $\mathring{p} \in T_Y X$.

2.14. Definition. We set

$$\widehat{\mathscr{D}}^{\nu}_{T_Y X \to Y} := H^n_{T_Y X}(\nu_{Y\times Y}(\mathscr{O}^{(0,n)}_{\Delta_{X/Y}})) = \boldsymbol{R}\Gamma_{T_Y X}(\nu_{Y\times Y}(\mathscr{O}^{(0,n)}_{\Delta_{X/Y}}))[n].$$

Then $\widehat{\mathscr{D}}^{\nu}_{T_Y X\to Y}$ is a $(\widehat{\mathscr{D}}^{\nu}_{T_Y X}\otimes \tau_Y^{-1}(\mathscr{D}^{\infty}_Y)^{\mathrm{op}})$-Module, and under an admissible local coordinate system we have an exact sequence $0 \to \widehat{\mathscr{D}}^{\nu}_{T_Y X} \xrightarrow{\cdot\partial_\tau} \widehat{\mathscr{D}}^{\nu}_{T_Y X} \to \widehat{\mathscr{D}}^{\nu}_{T_Y X\to Y} \to 0$.

2.15. Remark. $\widehat{\mathscr{D}}^{\nu}_{T_Y X\to Y}|_Y = \mathscr{O}\widetilde{\mathscr{D}}_{Y|L}$ is defined by Oaku [18, Definition 2.3].

2.16. Definition. (1) For any $\mathscr{F} \in \mathbf{D}^{\mathrm{b}}(\widehat{\mathscr{D}}^{\nu}_{T_Y X})$, we set

$$\widehat{\Psi}_Y(\mathscr{F}) := \boldsymbol{R}\mathscr{H}om_{\widehat{\mathscr{D}}^{\nu}_{T_Y X}}(\widehat{\mathscr{D}}^{\nu}_{T_Y X\to Y}, \mathscr{F}).$$

Then $\widehat{\Psi}_Y(\mathscr{F})$ is represented by $\mathscr{F} \xrightarrow{\partial_\tau\cdot} \mathscr{F}$ under an admissible local coordinate system.

(2) For any $\mathscr{N} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, we set

$$\widehat{\Psi}^{\mathscr{D}}_Y(\mathscr{N}) := \widehat{\Psi}_Y(\widehat{\mathscr{D}}^{\nu}_{T_Y X}\underset{\tau^{-1}_{X,Y}\mathscr{D}_X}{\overset{L}{\otimes}}\tau^{-1}_{X,Y}\mathscr{N}), \quad \Psi^{\infty}_Y(\mathscr{N}) := \mathring{\sigma}^{-1}\widehat{\Psi}^{\mathscr{D}}_Y(\mathscr{N}).$$

2.17. Proposition. *Let $\mathscr{N} \in \mathfrak{Coh}(\mathscr{D}_X)$. Then $H^i\Psi^{\infty}_Y(\mathscr{N}) = 0$ holds for $i \notin [-n,1]$, and $\Psi^{\infty}_Y(\mathscr{N})$ is represented by a bounded complex of $\mathscr{D}^{\infty}_Y$-Modules.*

2.18. Example. (1) $\widehat{\Psi}_Y(\nu_Y(\mathscr{O}_X)) \simeq \tau_Y^{-1}\mathscr{O}_Y$.
(2) $\tau_Y^{-1}\mathscr{D}^{\infty}_Y \rightleftarrows \widehat{\Psi}_Y(\widehat{\mathscr{D}}^{\nu}_{T_Y X\to Y})$.
(3) $\Psi^{\infty}_Y(\mathscr{D}_X/\mathscr{D}_X\vartheta) \simeq \Psi^{\infty}_Y(\mathscr{D}_X/\mathscr{D}_X\partial_\tau) \simeq \mathscr{D}^{\infty}_Y$.
(4) If $\mathscr{M} \in \mathfrak{Coh}(\mathscr{D}_X)$ satisfies that $\operatorname{supp}\mathscr{M} \subset Y$, then $\Psi^{\infty}_Y(\mathscr{M}) = 0$.

§ 3. Holomorphic Solutions to Fuchsian Systems

We inherit the notation from the preceding section.

3.1. Theorem. *Let $P = \vartheta - A(z,\tau,\partial_z)$ be a Fuchsian Volevič system of size m. Then for any $\mathring{p} \in \dot{T}_Y X$, the following hold:*

$$\widehat{\Psi}^{\mathscr{D}}_Y(\mathscr{D}^m_X/\mathscr{D}^m_X P)_{\mathring{p}} \simeq \widehat{\Psi}_Y(\widehat{\mathscr{D}}^{\nu}_{T_Y X\to Y})^m_{\mathring{p}} \simeq (\mathscr{D}^{\infty}_{Y,\tau_Y(\mathring{p})})^m.$$

For the proof, we use the results of Tahara [22].

3.2. Proposition. (1) *If P is a Fuchsian operator of weight (m,w), then locally $\Psi^{\infty}_Y(\mathscr{D}_X/\mathscr{D}_X P) \simeq (\mathscr{D}^{\infty}_Y)^m$.*
(2) *If $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$, then $H^i\Psi^{\infty}_Y(\mathscr{M}) = 0$ holds for $i \notin [-n,0]$.*

3.3. Remark. Let $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$. Then $\Psi_Y^\infty(\mathscr{M})$ is represented by

$$0 \to (\mathscr{D}_Y^\infty)^{r_n}/(\mathscr{D}_Y^\infty)^{r_{n+1}}Q \to (\mathscr{D}_Y^\infty)^{r_{n-1}} \to \cdots \to (\mathscr{D}_Y^\infty)^{r_1} \to (\mathscr{D}_Y^\infty)^{r_0},$$

where $r_i \in \mathbb{N}$ and $Q \in \mathrm{Mat}_{r_{n+1},r_n}(\mathscr{D}_Y^\infty)$.

For any $\mathscr{L} \in \mathbf{D}^{\mathrm{b}}(\mathscr{D}_Y^\infty)$, we set $\boldsymbol{D}_Y^\infty \mathscr{L} := \boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_Y^\infty}(\mathscr{L}, \mathscr{D}_Y^\infty) \overset{\boldsymbol{L}}{\underset{\mathscr{O}_Y}{\otimes}} \Omega_Y^{\otimes -1}[n]$.

3.4. Proposition. *Let $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$. Then there exist the following the following isomorphisms*:

$$\Psi_Y^\infty(\boldsymbol{D}_X\mathscr{M}) = \boldsymbol{D}_Y^\infty \Psi_Y^\infty(\mathscr{M}), \quad \Psi_Y^\infty(\mathscr{M}) = \boldsymbol{D}_Y^\infty \Psi_Y^\infty(\boldsymbol{D}_Y\mathscr{M}).$$

3.5. Proposition. (1) *For any $\mathscr{N} \in \mathfrak{Coh}(\mathscr{D}_X)$, there exists a natural morphism*

$$\Psi_Y^\infty(\mathscr{N}) \to \mathscr{D}_Y^\infty \overset{\boldsymbol{L}}{\underset{\mathscr{D}_Y}{\otimes}} \boldsymbol{D}f^*\mathscr{N}.$$

(2) *For any $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$, there exists a natural morphism*

$$\mathscr{D}_Y^\infty \overset{\boldsymbol{L}}{\underset{\mathscr{D}_Y}{\otimes}} \boldsymbol{D}f^!\mathscr{M} \to \Psi_Y^\infty(\mathscr{M}).$$

As usual, $\mathscr{C}_{Y|X}^{\mathbb{R}} := H^1\mu_Y(\mathscr{O}_X) = \mu_Y(\mathscr{O}_X)[1]$ denotes the sheaf of *holomorphic microfunctions* on T_Y^*X. Then $\mathscr{B}_{Y|X}^\infty := \mathscr{C}_{Y|X}^{\mathbb{R}}|_Y = H_Y^1(\mathscr{O}_X) = \boldsymbol{R}\Gamma_Y(\mathscr{O}_X)[1]$ is the sheaf of *holomorphic hyperfunctions.*

3.6. Theorem. *For any $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$, there exist the following isomorphisms between distinguished triangles*:

$$\begin{array}{ccc}
f^{-1}\boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X) & = & \boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M}, \mathscr{O}_Y) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M}, \hat{\sigma}^{-1}\nu_Y(\mathscr{O}_X)) & = & \boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_Y^\infty}(\Psi_Y^\infty(\mathscr{M}), \mathscr{O}_Y) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M}, {}^*\hat{\sigma}^{-1}\mathscr{C}_{Y|X}^{\mathbb{R}}) & = & \boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M}, {}^*\hat{\sigma}^{-1}\mathscr{C}_{Y|X}^{\mathbb{R}}), \\
\downarrow{+1} & & \downarrow{+1}
\end{array}$$

$$\begin{array}{ccc}
\boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_{Y|X}^\infty) & = & \boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_Y}(\boldsymbol{D}f^!\mathscr{M}, \mathscr{O}_Y)[-1] \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M}, {}^*\hat{\sigma}^{-1}\mathscr{C}_{Y|X}^{\mathbb{R}}) & = & \boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M}, {}^*\hat{\sigma}^{-1}\mathscr{C}_{Y|X}^{\mathbb{R}}) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M}, \hat{\sigma}^{-1}\nu_Y(\mathscr{O}_X)) & = & \boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_Y^\infty}(\Psi_Y^\infty(\mathscr{M}), \mathscr{O}_Y). \\
\downarrow{+1} & & \downarrow{+1}
\end{array}$$

3.7. Remark. Let $\mathscr{M} = \mathscr{D}_X/\mathscr{D}_X P$, where P is a Fuchsian partial differential operator of weight (m, w), or $\mathscr{M} = \mathscr{D}_X^m/\mathscr{D}_X^m P$, where P is a Fuchsian Volevič system of size m. Then locally $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \hat{\sigma}^{-1}\nu_Y(\mathscr{O}_X)) \simeq \mathscr{O}_Y^{\oplus m}$ (see Mandai [12] or Mandai-Tahara [13]).

Let $\mathcal{R}_Y(\mathscr{D}_X)$ be the subcategory of $\mathfrak{Coh}(\mathscr{D}_X)$ consisting of regular-specializable $\mathscr{D}_X$-Modules, and $\Psi_Y(\mathscr{M})$ (resp. $\Phi_Y(\mathscr{M})$) denotes the nearby cycle (resp. the vanishing cycle) of $\mathscr{M}$. We remark that $\mathscr{M} \in \mathcal{R}_Y(\mathscr{D}_X)$ if and only if the following holds: for any $u \in \mathscr{M}$, locally there exists $P \in \mathscr{D}_X$ such that $Pu = 0$, where P is of the following form:

$$P = \vartheta^m + \sum_{i=0}^{m-1} b_i \vartheta^i + \tau \sum_{|\alpha|+i \leqslant m} a_{\alpha,i}(z,\tau)\partial_z^\alpha \vartheta^i \quad (b_i \in \mathbb{C}).$$

For any $\mathscr{M} \in \mathcal{R}_Y(\mathscr{D}_X)$, we have the following distinguished triangles (see [9]):

$$\begin{array}{ccc}
f^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X) & = & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M}, \mathscr{O}_Y) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \hat{\sigma}^{-1}\nu_Y(\mathscr{O}_X)) & = & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\Psi_Y(\mathscr{M}), \mathscr{O}_Y) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, {}^*\hat{\sigma}^{-1}\mathscr{C}^{\mathbb{R}}_{Y|X}) & = & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\Phi_Y(\mathscr{M}), \mathscr{O}_Y), \\
\downarrow{+1} & & \downarrow{+1}
\end{array}$$

$$\begin{array}{ccc}
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}^{\infty}_{Y|X}) & = & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^!\mathscr{M}, \mathscr{O}_Y)[-1] \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, {}^*\hat{\sigma}^{-1}\mathscr{C}^{\mathbb{R}}_{Y|X}) & = & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\Phi_Y(\mathscr{M}), \mathscr{O}_Y) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \hat{\sigma}^{-1}\nu_Y(\mathscr{O}_X)) & = & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\Psi_Y(\mathscr{M}), \mathscr{O}_Y). \\
\downarrow{+1} & & \downarrow{+1}
\end{array}$$

3.8. Theorem. *If $\mathscr{M} \in \mathcal{R}_Y(\mathscr{D}_X)$, then $\Psi_Y^\infty(\mathscr{M}) \simeq \mathscr{D}_Y^\infty \overset{\boldsymbol{L}}{\underset{\mathscr{D}_Y}{\otimes}} \Psi_Y(\mathscr{M})$. In particular, if Y is non-characteristic for $\mathscr{M}$, then $\Psi_Y^\infty(\mathscr{M}) \simeq \mathscr{D}_Y^\infty \overset{\boldsymbol{L}}{\underset{\mathscr{D}_Y}{\otimes}} \boldsymbol{D}f^*\mathscr{M}$.*

§ 4. Boundary Values for Hyperfunction Solutions

We denote by $\mathscr{B}_M$ and $\mathscr{C}_M$ the sheaves of *hyperfunctions* on M and of *microfunctions* on T_M^*X respectively.

4.1. Definition ([4], [5]). We define the sheaf on $\sqrt{-1}\,T_N^*M$ of second hyperfunctions by

$$\mathscr{B}^2_{\sqrt{-1}T_N^*M} := H^{n+1}_{\sqrt{-1}T_N^*M}(\mathscr{C}^{\mathbb{R}}_{Y|X}) \otimes \mathscr{or}_{N/Y} \simeq \boldsymbol{R}\Gamma_{\sqrt{-1}T_N^*M}(\mu_Y(\mathscr{O}_X)) \otimes \mathscr{or}_{N/Y}[n+2].$$

By Holmgren type theorem for hyperfunctions and [4], [5], we have monomorphisms

$$\Gamma_{M_+}(\mathscr{B}_M)|_N \rightarrowtail {}^*\hat{s}^{-1}\mathscr{C}_M \rightarrowtail {}^*\hat{s}^{-1}\mathscr{B}^2_{\sqrt{-1}T^*_N M}.$$

Hence we obtain

4.2. Theorem. *Let $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$. Then there exists the following morphism between distinguished triangles*:

$$\begin{array}{ccc}
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_N(\mathscr{B}_M)) \otimes or_{N/M} & = & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^!\mathscr{M}, \mathscr{B}_N)[-1] \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{M_+}(\mathscr{B}_M))|_N \otimes or_{N/M} & \to & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, {}^*\hat{s}^{-1}\mathscr{B}^2_{\sqrt{-1}T^*_N M}) \otimes or_{N/M} \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{\Omega_+}(\mathscr{B}_M))|_N & \longrightarrow & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y^\infty}(\Psi_Y^\infty(\mathscr{M}), \mathscr{B}_N). \\
\downarrow{+1} & & \downarrow{+1}
\end{array}$$

4.3. Definition. Let $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$. By Theorem 4.2 we can define

$$\gamma_+ : \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{\Omega_+}(\mathscr{B}_M))|_N \to \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y^\infty}(\Psi_Y^\infty(\mathscr{M}), \mathscr{B}_N). \tag{4.1}$$

Taking cohomologies, we have

4.4. Proposition. *Let $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$. Then* (4.1) *induces a monomorphism*

$$\gamma_+^0 : \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{\Omega_+}(\mathscr{B}_M))|_N \rightarrowtail \mathscr{H}om_{\mathscr{D}_Y^\infty}(H^0\Psi_Y^\infty(\mathscr{M}), \mathscr{B}_N).$$

Next, we recall definitions of several sheaves attached to the boundary due to Oaku [18]. Note that in Oaku [18] these sheaves are defined on cosphere bundles, so we shall present definitions on cotangent bundles along the line of Oaku-Yamazaki [19]. Although only the higher-codimensional case is treated in [19], the same arguments also work in the one-codimensional case.

4.5. Definition. We set:

$$\begin{aligned}
\mathscr{C}_{N|M} &:= s_{L\pi}^{-1}\,\mu_{\widetilde{M}_N}(\boldsymbol{R}\Gamma_{\Omega_L}(p_L^{-1}\boldsymbol{R}\Gamma_L(\mathscr{O}_X))) \otimes or_{M/X}[n], \\
\widetilde{\mathscr{C}}_{N|M} &:= \mu_{T_N M}(\nu_Y(\boldsymbol{R}\Gamma_L(\mathscr{O}_X))) \otimes or_{M/X}[n], \\
\widetilde{\mathscr{B}}_{N|M} &:= \widetilde{\mathscr{C}}_{N|M}|_{T_N M}.
\end{aligned}$$

Then $\mathscr{C}_{N|M}$ and $\widetilde{\mathscr{C}}_{N|M}$ are concentrated in degree zero, and $\nu_N(\mathscr{B}_M) = \mathscr{C}_{N|M}|_{T_N M}$.

4.6. Proposition ([18]). (1) *$\mathscr{C}_{N|M}$ and $\widetilde{\mathscr{C}}_{N|M}$ are concentrated in degree zero; that is, $\widetilde{\mathscr{C}}_{N|M}$ and $\mathscr{C}_{N|M}$ are regarded as sheaves on $T^*_{T_N M}T_Y L$.*

(2) *There exists a canonical monomorphism* $s^*_{N|M} \colon \mathscr{C}_{N|M} \to \widetilde{\mathscr{C}}_{N|M}$.

(3) $\nu_N(\mathscr{B}_M) = \mathscr{C}_{N|M}|_{T_N M}$, *and there exists the following commutative diagram with exact rows on* $T_N M$:

$$\begin{array}{ccccccccc} 0 & \to & \nu_Y(\mathscr{B}\mathscr{O}_L)|_{T_N M} & \to & \nu_N(\mathscr{B}_M) & \to & \dot\pi_{N|M*}\mathscr{C}_{N|M} & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \nu_Y(\mathscr{B}\mathscr{O}_L)|_{T_N M} & \longrightarrow & \widetilde{\mathscr{B}}_{N|M} & \longrightarrow & \dot\pi_{N|M*}\widetilde{\mathscr{C}}_{N|M} & \to & 0. \end{array}$$

Here $\mathscr{B}\mathscr{O}_L := H^1_L(\mathscr{O}_X) \otimes \mathscr{or}_{L/X} \simeq \boldsymbol{R}\Gamma_L(\mathscr{O}_X) \otimes \mathscr{or}_{L/X}[1]$. *Note that* $\nu_Y(\mathscr{B}\mathscr{O}_L)$ *is concentrated in degree zero.*

4.7. Definition. Let $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$. Then we can define the morphism γ_+:

$$\begin{aligned} \gamma_+ \colon i^{-1}_{\pi+}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_{N|M}) &\to i^{-1}_{\pi+}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \widetilde{\mathscr{C}}_{N|M}) \\ &\simeq \tilde\tau^{-1}_{\pi+}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}^\infty_Y}(\Psi^\infty_Y(\mathscr{M}), \mathscr{C}_N). \end{aligned}$$

The restriction of γ_+ to the zero-section $T_N M^+$ of $T^*_{T_N M^+} T_Y L^+$ coincides with the boundary value morphism (4.1).

We can obtain the following Holmgren type theorem:

4.8. Theorem. *Let* $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$. *Then the morphism* γ_+ *gives a monomorphism*

$$\gamma^0_+ \colon i^{-1}_{\pi+}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_{N|M}) \rightarrowtail \tilde\tau^{-1}_{\pi+}\mathscr{H}om_{\mathscr{D}^\infty_Y}(H^0\Psi^\infty_Y(\mathscr{M}), \mathscr{C}_N).$$

4.9. Remark. Theorem 4.8 gives another proof of Proposition 4.4.

4.10. Theorem. *Let* $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$. *Assume that* $\mathscr{M}$ *is near-hyperbolic at* $\mathring{x} \in N$ *in the dt-codirection. Then, for any* $p^* = (\hat s(\mathring{x}); \sqrt{-1}\,\mathring{y}^*) \in T^*_{T_N M^+} T_Y L^+$, *there exists an isomorphism*

$$\gamma_+ \colon \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_{N|M})_{p^*} \simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}^\infty_Y}(\Psi^\infty_Y(\mathscr{M}), \mathscr{C}_N)_{p_0}.$$

Here $p_0 := \tilde\tau_\pi(p^*) = (\mathring{x}; \sqrt{-1}\,\mathring{y}^*) \in T^*_N Y$. *In particular, there exists an isomorphism*

$$\begin{aligned} \gamma_+ \colon \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{\Omega_+}(\mathscr{B}_M))_{\mathring{x}} &= \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \nu_N(\mathscr{B}_M))_{\hat s(\mathring{x})} \\ &\simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}^\infty_Y}(\Psi^\infty_Y(\mathscr{M}), \mathscr{B}_N)_{\mathring{x}}. \end{aligned}$$

We consider the mappings:

$$\begin{array}{ccccc} T^*_M X & \xleftarrow{f_{N\pi}} & N \underset{M}{\times} T^*_M X & \xrightarrow{f_{Nd}} & T^*_N Y \\ {\scriptstyle k}\big\downarrow & \square & {\scriptstyle k}\big\downarrow & \square & {\scriptstyle k}\big\downarrow \\ T^*X & \xleftarrow{f_\pi} & Y \underset{X}{\times} T^*X & \xrightarrow{f_d} & T^*Y. \end{array}$$

Then the sheaf of *microfunction with a real analytic parameter* t on $T^*_N Y$ is defined by

$$\mathscr{C}^A_{N|M} := f_{Nd!}\, f^{-1}_{N\pi}\mathscr{C}_M \simeq H^{n+1}(k^{-1}\boldsymbol{R}f_{d!}\, f^{-1}_{\pi}\mu\mathit{hom}(\mathbb{C}_M, \mathscr{O}_X) \otimes \mathit{or}_{M/X}).$$

The sheaf $\mathring{\mathscr{C}}_{N|M_+}$ of *mild microfunctions* on $T^*_N Y$ is defined by Kataoka [8], and reformulated by Schapira-Zampieri as [21]

$$\mathring{\mathscr{C}}_{N|M_+} = H^{n+1}(\boldsymbol{R}f_{d!}\, f^{-1}_{\pi}\mu\mathit{hom}(\mathbb{C}_{\Omega_+}, \mathscr{O}_X) \otimes \mathit{or}_{M/X}).$$

Then we have natural monomorphisms ([17], [19]):

$$\tilde{\tau}^{-1}_{\pi+}\mathring{\mathscr{C}}^A_{N|M_+} \hookrightarrow \tilde{\tau}^{-1}_{\pi+}\mathring{\mathscr{C}}_{N|M_+} \hookrightarrow i^{-1}_{\pi+}\mathscr{C}_{N|M},$$

and restricting to N, we have natural monomorphisms

$$\mathscr{B}^A_{N|M} \hookrightarrow \mathring{\mathscr{B}}_{N|M_+} \hookrightarrow \hat{s}^{-1}\nu_N(\mathscr{B}_M) = \Gamma_{\Omega_+}(\mathscr{B}_M)|_N .$$

Here $\mathring{\mathscr{B}}_{N|M_+}$ denotes the sheaf of mild hyperfunctions. Setting $Df^*\mathscr{M} := H^0\boldsymbol{D}f^*\mathscr{M}$, we can obtain a monomorphism

$$\mathscr{H}om_{\mathscr{D}_Y}(Df^*\mathscr{M}, \mathscr{C}_N) \rightarrowtail \mathscr{H}om_{\mathscr{D}^\infty_Y}(H^0\Psi^\infty_Y(\mathscr{M}), \mathscr{C}_N).$$

For any $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$, by construction and [24], we obtain the following:

(1) There exist the following commutative diagrams:

$$\begin{tikzcd}
\tilde{\tau}^{-1}_{\pi+}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}^A_{N|M}) \arrow[d] \arrow[dr, bend left, "\gamma^A"] & \\
\tilde{\tau}^{-1}_{\pi+}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathring{\mathscr{C}}_{N|M_+}) \arrow[r, "\mathring{\gamma}"] \arrow[d] & \tilde{\tau}^{-1}_{\pi+}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M}, \mathscr{C}_N) \arrow[d] \\
i^{-1}_{\pi+}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_{N|M}) \arrow[r, "\gamma_+"] & \tilde{\tau}^{-1}_{\pi+}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}^\infty_Y}(\Psi^\infty_Y(\mathscr{M}), \mathscr{C}_N)
\end{tikzcd} \tag{4.2}$$

$$\begin{tikzcd}
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}^A_{N|M}) \arrow[d] \arrow[dr, bend left, "\gamma^A"] & \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathring{\mathscr{B}}_{N|M_+}) \arrow[r, "\mathring{\gamma}"] \arrow[d] & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M}, \mathscr{B}_N) \arrow[d] \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{\Omega_+}(\mathscr{B}_M))|_N \arrow[r, "\gamma_+"] & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}^\infty_Y}(\Psi^\infty_Y(\mathscr{M}), \mathscr{B}_N).
\end{tikzcd} \tag{4.3}$$

Moreover (4.2) and (4.3) induce the following monomorphisms:

$$\begin{tikzcd}
\tilde{\tau}^{-1}_{\pi+}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}^A_{N|M}) \arrow[d, tail] \arrow[dr, tail, bend left, "\gamma^{A,0}"] & \\
\tilde{\tau}^{-1}_{\pi+}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathring{\mathscr{C}}_{N|M_+}) \arrow[r, tail, "\mathring{\gamma}^0"] \arrow[d, tail] & \tilde{\tau}^{-1}_{\pi+}\mathscr{H}om_{\mathscr{D}_Y}(Df^*\mathscr{M}, \mathscr{C}_N) \arrow[d, tail] \\
i^{-1}_{\pi+}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_{N|M}) \arrow[r, tail, "\gamma^0_+"] & \tilde{\tau}^{-1}_{\pi+}\mathscr{H}om_{\mathscr{D}^\infty_Y}(H^0\Psi^\infty_Y(\mathscr{M}), \mathscr{C}_N)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}^A_{N|M}) \arrow[d, tail] \arrow[rr, tail, bend left, "\gamma^{A,0}"] & & \\
\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathring{\mathcal{B}}_{N|M_+}) \arrow[d, tail] \arrow[rr, tail, "\mathring{\gamma}^0"] & & \mathcal{H}om_{\mathcal{D}_Y}(Df^*\mathcal{M}, \mathcal{B}_N) \arrow[d, tail] \\
\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{\Omega_+}(\mathcal{B}_M))|_N \arrow[rr, tail, "\gamma^0_+"] & & \mathcal{H}om_{\mathcal{D}^\infty_Y}(H^0\Psi^\infty_Y(\mathcal{M}), \mathcal{B}_N).
\end{tikzcd}$$

(2) Let $p^* = (\hat{s}(\mathring{x}); \sqrt{-1}\,\mathring{y}^*) \in T^*_{T_N M^+}\dot{T}_Y L^+$. Assume that $\mathcal{M}$ is near-hyperbolic at $\mathring{x} \in N$ in the $\pm dt$-codirections. Then γ^A, $\mathring{\gamma}$ and γ_+ are isomorphisms at p^* in (4.2) (resp. at $\mathring{x}$ in (4.3)).

4.11. Example. Consider $P = \prod_{i=1}^{m}(\vartheta - \alpha_i(x))^{\nu_i}$ such that $\alpha_i(0)$, $\alpha_i(0) - \alpha_j(0) \notin \mathbb{Z}$ $(i \neq j)$. Then $u(x,t) \in \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{D}_X P, \Gamma_{\Omega_+}(\mathcal{B}_M))_0$ is written as

$$u(x,t) = \sum_{i=1}^{m}\sum_{j=1}^{\nu_i} u_{ij}(x)\, t^{\alpha_i(x)}(\log t)^{j-1},$$

and $\gamma^0_+(u) = \{u_{ij}(x);\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant \nu_i\}$. Futher $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{D}_X P, \mathcal{B}^A_{N|M})_0 = 0$.

4.12. Example. Assume $n = 1$ (hence $x \in N = \mathbb{R}$). For any $P \in \mathcal{D}_X$, we set $\mathcal{M}_P := \mathcal{D}_X/\mathcal{D}_X P$.

(1) Let $P := \vartheta - i - x$ $(i \in \mathbb{N})$ and $u(x,t) \in \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_P, \Gamma_{\Omega_+}(\mathcal{B}_M))_0$. Then we have $\Psi^\infty_Y(\mathcal{M}_P) \simeq \mathcal{D}^\infty_Y$, $u(x,t) = u_0(x)t^{i+x}$, and $\gamma^0_+(u) = u_0(x)$. In addition if $u(x,t) \in \mathcal{B}^A_{N|M,0}$, we have $xu_0(x) = 0$, hence $xu_0(x) = C\delta(x)$, where $C \in \mathbb{C}$. In this case we have $C\delta(x)t^{i+x} = C\delta(x)t^i$, and $\gamma^{A,0}(u) = C\delta(x)$.

(2) Let $P := (\vartheta - \alpha_1)(\vartheta - \alpha_2) - xt\vartheta$ and $u(x,t) \in \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_P, \Gamma_{\Omega_+}(\mathcal{B}_M))_0$.

(i) If $(\alpha_1, \alpha_2) = (-1, 0)$, we have

$$u(x,t) = u_{-1}(x)\Big(\frac{1}{t} - x\sum_{i=1}^{\infty}\frac{(xt)^i}{i\,(i+1)!} - x\log(t + \sqrt{-1}\,0)\Big) + u_0(x),$$

and $\gamma^0_+(u) = \{u_{-1}(x), u_0(x)\}$. In addition if $u(x,t) \in \mathcal{B}^A_{N|M,0}$, we have $u_{-1}(x) = 0$ and $\gamma^{A,0}(u) = u_0(x)$.

(ii) If $(\alpha_1, \alpha_2) = (0, 1)$, we have

$$u(x,t) = u_0(x) + u_1(x)\frac{e^{xt} - 1}{x},$$

and $u(x,t) \in \mathcal{B}^A_{N|M,0}$, hence $\gamma^0_+(u) = \gamma^{A,0}(u) = \{u_0(x), u_1(x)\}$. Note that in this case, we have $P = t^2(\partial_t^2 - x\partial_t)$, and Y is non-characteristic for $\partial_t^2 - x\partial_t$.

(iii) If $(\alpha_1, \alpha_2) = (1, 1)$, we have

$$u(x,t) = u_0(x)\,e^{xt}t - u_1(x)\,t\Big(\sum_{i=1}^{\infty}\sum_{j=1}^{i}\frac{(xt)^i}{i!j} + u_1(x)\,e^{xt}\log(t + \sqrt{-1}\,0)\Big),$$

and $\gamma_+^0(u) = \{u_0(x), u_1(x)\}$. In addition if $u(x,t) \in \mathscr{B}^A_{N|M,0}$, we have $u_1(x) = 0$, and $\gamma^{A,0}(u) = u_0(x)$.

(iv) If $(\alpha_1, \alpha_2) = (1,2)$, we have

$$u(x,t) = u_1(x)\, t\Big(1 - \sum_{i=1}^{\infty}\sum_{j=1}^{i} \frac{(xt)^{i+1}}{i!j} + e^{xt}xt\log(t+\sqrt{-1}\,0)\Big) + u_2(x)e^{xt}t^2,$$

and $\gamma_+^0(u) = \{u_1(x), u_2(x)\}$. In addition if $u(x,t) \in \mathscr{B}^A_{N|M,0}$, we have $xu_1(x) = 0$, hence $u_1(x) = C\delta(x)$. Thus

$$u(x,t) = C\delta(x)\, t + u_2(x)e^{xt}t^2,$$

and $\gamma^{A,0}(u) = \{C\delta(x), u_2(x)\}$.

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